Asymptotic normalization coefficients of resonant and bound states from the phase shifts for $\alpha\alpha$ and α^{12}C scattering.

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Recently we have published a paper [Irgaziev, Phys. Rev. C 91, 024002 (2015)] where the S-matrix pole method (SMP) which is only valid for resonances has been developed to derive an explicit expression for the asymptotic normalization coefficient (ANC), and is applied to the low-energy resonant states of nucleon+α and $\alpha+^{12}$ C systems. The SMP results are compared with the effective-range expansion method (EFE) results. In the present paper the SMP and EFE plus the Padé-approximation are applied to study the excited 2^+ resonant states of ^{8}Be. A contradiction is found between descriptions of the experimental phase shift data for $\alpha\alpha$ scattering and of the ^{8}Be resonant energy for 2^+ state. Using the EFE method, we also calculate the ANC for the ^{8}Be ground 0^+ state with a very small width. This ANC agrees well with the value calculated using the known analytical expression for narrow resonances. In addition, for the $\alpha+^{12}$ C states 1^- and 3^- the SMP results are compared with the Padé approximation results. We find that the Padé-approximation improves a resonance width description compared with the EFE results. The EFE method is also used to calculate the ANCs for the bound ^{16}O ground 0^+ state and for the excited 1^- and 2^+ levels which are situated near the threshold of $\alpha+^{12}$ C channel.

I. INTRODUCTION

It is well-known that nuclear reactions at very low energies are key nuclear processes in stellar nucleosynthesis [1–4]. These reactions can occur as direct processes or through the formation of resonance. A direct reaction near the threshold is suppressed due to the Coulomb barrier, and the reaction through resonance therefore becomes critical when it is near the threshold. Examples of certain resonances of interest to nuclear astrophysics are presented below. In Ref. [5] we developed the S-matrix pole method (SMP) to derive an explicit expression for the asymptotic normalization coefficient (ANC) of a resonant Gamow radial wave function in the presence of the Coulomb interaction. We used an analytical S-matrix approximation in the form of a series of powers of the relative momentum k for the nonresonant part of the phase shift for the arbitrary orbital momentum l of colliding particles. Earlier in Ref. [6], this method was applied to calculate the resonance pole energy E_r and width Γ. In the earlier paper [5] we successfully applied the SMP to the resonances of the nucleon+α and $\alpha+^{12}$C systems. We obtained respective ANC values for different states of the nuclei ^{5}He,^{5} Li, and ^{16}O . We found that the Γ value calculated by the SMP agrees better with the other values from the literature, while the effective-range expansion method (EFE) overestimates Γ.

In the present paper we apply low-energy approaches (SMP, EFE and Padé-approximant) to the another nucleus ^{8}Be which is unstable even in the ground 0^+ state.

Hoyle predicted the existence of the resonance state of the ^{12}C nucleus with an excitation energy of 7.68 MeV [7] even before the actual observation of 7.65 MeV in experiments. Salpeter accepted Hoyle's idea and theoretically considered the ^{12}C creation mechanism as the result of a three α particles fusion with the intermediate creation of the narrow resonance ^{8}Be in the ground state. Fowler and his group carried out corresponding experiments which confirmed Hoyle's prediction. At the end of the life of red giant stars compressed by the gravitation, the temperature increases up to values $T > 10^8$ K. At such temperatures carbon creation occurs due to the two consecutive processes: $\alpha+\alpha \rightarrow {}^8$Be (0^+, ground state) and $\alpha+{}^8$Be $\rightarrow {}^{12}$C* (0^+, 7.65 MeV). The small difference ($\cong$ 0.28 MeV) between the ^{12}C energy level and those of the system $\alpha+{}^8$Be is especially important.

The EFE is also used in the present paper to calculate ANCs for the bound states of the system $\alpha+{}^{12}$C. Besides, we find that it is possible to improve an agreement for the resonance width Γ when a Padé-approximant is used instead of the EFE. In our previous paper [5] we study ^{16}O resonances in α^{12}C scattering. However, the properties of the ^{16}O bound states are also studied here because they are quite important for astrophysics.

The ^{12}C$(\alpha,\gamma)^{16}$O reaction is considered one of the key nuclear processes at the stage of helium combustion in stars (formation of the red giant). This reaction determines the relative content of ^{12}C/^{16}O in the process of stellar helium combustion. It directly affects the sequence and peculiarities of further combustion stages in massive stars, such as carbon combustion. One needs to know the ANC for the decay ^{16}O $\rightarrow \alpha+{}^{12}$C for different final states to find the rate of this radiative capture reaction. The main goals of the present paper are the applications of low-energy approaches to the other nucleus

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^{8}Be and to the ^{16}O states which are of interest to astrophysics but are not investigated in Ref. [5]. The article is organized as follows. In Sec. II we present main formulas of the S-matrix pole method. In Sec. III we study the properties of the ground ^{8}Be s wave state resonance in the $\alpha\alpha$ scattering. Short history of this level complex energy studying in the literature is given as well as our results for the energy, the nuclear vertex ($^8\mathrm{Be} \to \alpha + \alpha$) constant (NVC) and the ANC calculated using the EFE method. We have to use here the EFE instead of the SMP because this state situated just near the zero energy where the phase shift has essential singularity (see Eq. (20) in Ref. [5]). In Sec. IV we use all the three methods (SMP, EFE, and Padé) to calculate the excited d wave state resonance in the $\alpha\alpha$ scattering. We compare the SMP approximation results for the same excited 2^+ resonant state of ^{8}Be for the two variants when fitting the experimental phase shift data: taking the experimental resonance energy E_2 as fixed and considering E_2 as an additional fitting quantity. The latter leads to a very good agreement with the experimental phase shift data. It is shown that there is disagreement between the experimental energy dependence of the phase shift on the one hand and the resonance complex energy on the other. In Sec. V we consider the ^{16}O bound states: the ground 0^+ state as well as the excited bound states 1^- and 2^+. These states are very important for astrophysics. In Sec. VI we study the ^{16}O resonant states 1^- and 3^-. with bigger widths, which was recently considered in Ref. [5] where the very good description of the experimental phase shift data was achieved. We show that the Padé approximant describes width better than EFE. In Sec. VII (conclusion) the results of the present paper are discussed. For the bound ^{16}O excited state 2^+ our calculated ANC is compared with the one published in the literature.

II. THE MAIN FORMULAS OF THE S-MATRIX POLE METHOD (SMP)

All the SMP formulas needed to calculate the resonance energy, nuclear vertex constant (NVC) and asymptotic normalization coefficient (ANC) are given in Ref. [5]. Some of them are given below. [1] In the single-channel elastic scattering case the partial S-matrix element (without the pure Coulomb part $e^{i2\sigma_l} = \Gamma(l+1+i\eta)/\Gamma(l+1-i\eta)$ where $\Gamma(x)$ is the gamma-function, $\eta = z_1 z_2 \mu\alpha/k$ is the Sommerfeld parameter, α is the fine-structure constant, and μ is the reduced mass of the colliding nuclei with the charge numbers z_1 and z_2) is denoted as

$$S_l(k) = e^{i2\delta_l}. \tag{1}$$

Near an isolated resonance it can be approximated as [8]

$$S_l(k) = e^{2i\nu_l(k)} \frac{(k+k_r)(k-k_r^\star)}{(k-k_r)(k+k_r^\star)}, \tag{2}$$

where $k_r = k_0 - ik_i$ is the complex wave number of a resonance ($k_0 > k_i > 0$, and the symbol (*) means the complex conjugate operation). Using Eq. (2), one can rewrite Eq. (1) in the form

$$S_l(k) = e^{2i(\nu_l + \delta_r + \delta_a)}, \tag{3}$$

where $\delta_r = -\arctan\frac{k_i}{k-k_0}$ stands for the resonance phase shift, while $\delta_a = -\arctan\frac{k_i}{k+k_0}$ is an additional phase shift which contributes to the whole scattering phase shift. Thus the total phase shift is

$$\delta_l = \nu_l + \delta_r + \delta_a. \tag{4}$$

The partial scattering nonresonant phase shift $\nu_l(k)$ is a smooth function near the pole of the S-matrix element, corresponding to the resonance. The S-matrix element defined by Eq. (2) fulfills the conditions of analyticity, unitarity, and symmetry. Therefore we can expand $\nu_l(k)$ to a series

$$\nu_l(k) = \sum_{n=0}^{\infty} c_n (k-k_s)^n \tag{5}$$

in the vicinity of the pole corresponding to the resonance. The point k_s denotes a centered point, and a convergence radius should be shorter than the distance from the centered point to the closest singular point. The expansion coefficients c_n of Eq. (5) as well as k_0 and k_i (i.e., a resonance complex energy value) are determined by fitting the experimental values of the elastic scattering phase shifts δ_l given by Eq. (4). It is enough for the $\alpha\alpha$ system to limit the expansion (5) up to $n = 4$. The nonresonant phase shift $\nu_l(k)$ is an analytical function excluding the origin. In Ref. [9], the authors present the behavior of $\delta_l(k)$ near the origin as

$$\delta_l(k) = -\frac{2\pi}{(l!)^2} k^{2l+1} \eta^{2l+1} a_l e^{-2\pi\eta}, \tag{6}$$

where a_l is the scattering length for colliding nuclei. [2] The point $k = 0$ is an essential singularity point of the scattering phase shift. However, as a function of the momentum k, ν has normal analytical properties near the point corresponding to the resonance. Besides, the convergence region of Eq. (5) is limited due to the presence of an exchange Feynman diagram for the elastic scattering, leading to the logarithmic singularity which is absent in our models. The renormalized partial amplitude

[1] Here and below we use the unit system $\hbar = c = 1$.

[2] There is a misprint in our paper [5]: In the formula above ν was written instead of δ [9] but it does not matter because δ_r and δ_a do not have such singularity.

is constructed (see Refs [10–12]) for its analytical continuation to a resonance or bound-state energy region. According to its definition, the nuclear renormalized vertex constant $\tilde{G}_l$ (NVC) [13], NVC can be written as

$$\tilde{G}_l^2 = \frac{2\pi}{\mu^2}\frac{k_r k_i e^{i2\nu_l(k_r)}}{k_0\rho_l(k_r)} = \frac{\pi\Gamma}{\mu k_0}\frac{(1-ik_i/k_0)e^{i2\nu_l(k_r)}}{\rho_l(k_r)}. \quad (7)$$

where ρ_l is equal to

$$\rho_l(k) = \frac{2\pi\eta}{e^{2\pi\eta}-1}\prod_{n=1}^{l}\Big(1+\frac{\eta^2}{n^2}\Big). \quad (8)$$

Using the relationship between NVC $\tilde{G}_l$ and ANC C_l [13], we obtain

$$C_l = \frac{i^{-l}\mu}{\sqrt{\pi}}\frac{\Gamma(l+1+i\eta_r)}{l!}e^{-\frac{\pi\eta_r}{2}}\tilde{G}_l = i^{-l}\sqrt{\frac{\mu\Gamma}{k_0}}e^{-\frac{\pi\eta_r}{2}}\frac{\Gamma(l+1+i\eta_r)}{l!} \times e^{i\nu_l(k_r)}\sqrt{(1-ik_i/k_0)/\rho_l(k_r)}. \quad (9)$$

In the limit of a small width Γ, we obtain the formula

$$C_l^a = \sqrt{\frac{\mu\Gamma}{k_0}}e^{i(\nu_l(k_0)+\sigma_l(k_0)-\pi l/2)}, \quad (10)$$

which coincides with the expression derived in Ref. [14]. This formula (10) can be used to check the calculation results. All the necessary expressions for the EFE are published in the literature (see Ref. [5] and references therein).

III. THE GROUND ^{8}Be s WAVE STATE RESONANCE IN THE $\alpha\,\alpha$ SCATTERING

In the present paper we continue to study resonances for light nuclei. We consider the nucleus ^{8}Be which is not bound in the ground state due to the Coulomb repulsion between α-particles. This state presents a very narrow resonance with the pole at the center-of-mass system (c.m.s.) energy (see the review [15] and the references therein):

$$E_\alpha = E_0 - i\Gamma_0/2,\ E_0 = 91.84\,\text{keV},\ \Gamma_0 = 5.57\pm0.25\,\text{eV}.$$

The α-particle model is a good approximation for a description of ^{8}Be characteristics because of the ^{4}He nucleus large binding energy. The other channels have thresholds situated at $E_{lab} > 35$ MeV. The Q value for the reaction $\alpha+\alpha \to {}^8$Be has changed over time in the literature. The value $Q = 94.5 \pm 1.5$ keV is found in Ref. [16]. Fowler's experimental group obtains $Q = 93.7 \pm 0.9$ keV [17]. In Ref. [18] the results of the phase shift analysis (see references in Ref. [18]) are used to find the ^{8}Be resonance parameters by applying the EFE. In Ref. [18] the values of the energy E_0 and the width Γ_0 for narrow resonance of ^{8}Be in the ground state in the cms frame are as follow:

$$E_0 = 94.5 \pm 1.4\,\text{keV},\ \ \Gamma_0 = 4.5 \pm 3\ \text{eV}.$$

The E_0 value is determined in Ref. [18] as the energy when the phase shift δ_0 passes $\pi/2$ and the resonance width Γ_0 is obtained from the equation expressed Γ in terms of the rate of changing δ_0 in the resonance region ($\pi/4 < \delta_0 < 3\pi/4$). The nuclear interaction is revealed in the scattering cross section at energy $E_0 > 300$ keV, i.e., after the resonance region where the s-wave phase shift δ_0 jumps from 0 up almost to π. For an analytical continuation of the cross section into the resonance region, the authors of Ref. [18] apply the EFE with the Coulomb interaction taken into account, using the formula by Landau and Smorodinsky (see Ref. [19] which is valid in the physical energy region and the reference to the original paper). In Ref. [18] the effective-range function $K_l(k^2)$ is expanded in a series over k^2 up to power of k^4:

$$K_l(k^2) = -1/a + (r/2)k^2 - Pr^3k^4, \quad (11)$$

or an equivalent expansion in a series over E_α

$$K_l(E_\alpha) = a_0 + a_1E_\alpha + a_2E_\alpha^2, \quad (12)$$

which adequately describes the experimental values of δ_0 at the cms energy $E_\alpha \cong 2.5$ MeV. Later in Ref. [20] an uncertainty of the measured cross-section from the Coulomb (Mott) cross-section is found experimentally even in the resonance region. As in Ref. [18], the scattering of the singly-charged ion ^{4}He$^+$ on the neutral ^{4}He atoms is investigated. The following results for E_0 and Γ_0 are obtained in Ref. [20]:

$$E_0 = 92.12 \pm 0.05\ \text{keV},\ \ \Gamma_0 = 6.8 \pm 1.7\ \text{eV}. \quad (13)$$

In later experiments (see [18]), the E_0 and Γ_0 values do not change much compared with (13), but their uncertainties appreciably decrease. The scattering amplitude is defined in [20] as the sum of the Coulomb and the nuclear amplitudes. The nuclear amplitude is taken in the Breit-Wigner form, which may be a reasonable approximation for a narrow resonance.

In our paper we use the EFE, Padé approximant, and SMP. The last method is used for defining a resonant energy, including that of broad resonances [6]. In the case of a narrow resonance, its energy depends less on the used method. We show below that our result for the ^{8}Be ground state energy is in good agreement with (Eq. 13). As input data, we use the phase shift borrowed from Ref. [21] (see table II in Ref. [21], p. 252). We use the following values for the resonance energy and width (see Ref. [22])

$$E_0 = 91.84\ \text{keV};\ \ \Gamma_0 = 5.57\ \text{eV}. \quad (14)$$

In Fig. 1 we show the results of fitting the effective-

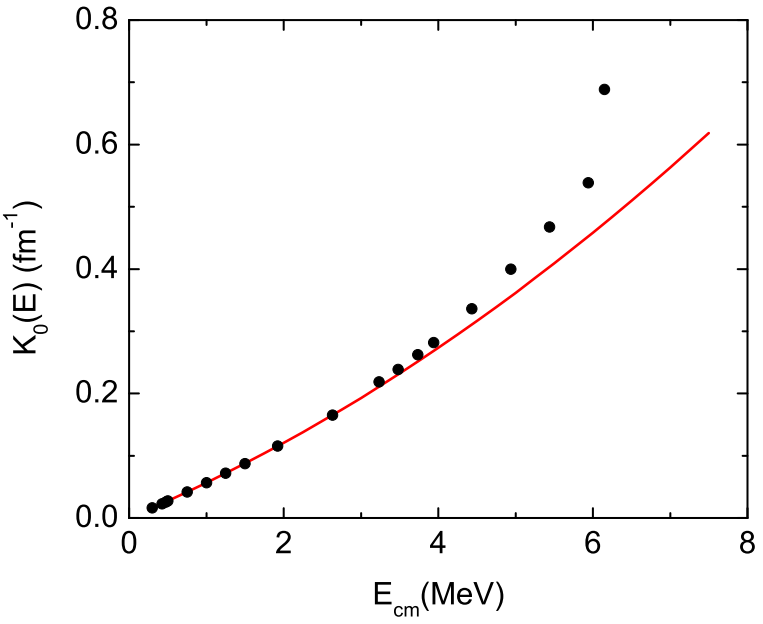

FIG. 1. Comparison of the fitted effective-range function for the s wave $\alpha\alpha$ elastic scattering with the experimental values. The experimental data are taken from Ref. [21]. The energy is given in the c.m.s. frame.

range function dependence on energy for the s state of ^{8}Be using the phase shift table given in Ref. [21]. It was indicated earlier in Ref. [23] that the radius of convergence of the expansions in Eq. (11) around zero energy is determined by the singularity of the two-pion-exchange Feynman diagram at an energy of $E_{sing} \approx 5$ MeV (more precisely, at 4.76 MeV), since one-pion exchange is suppressed in isospin. The fitting curve corresponds to the effective-range function approximation (11) with the parameters:

$$a = -1724.1 \text{ fm}; \quad r = 1.0848 \text{ fm}; \quad P = -0.34717. \quad (15)$$

This set does not differ much from that given in Ref. [18]:

$$a = -1760 \text{ fm}; \quad r = 1.096 \text{ fm}; \quad P = -0.314. \quad (16)$$

We note that in Ref. [18], misprints are made in writing the shape parameter P: A dimension (cm) of P was shown which actually is a dimensionless quantity, and the wrong sign $P = +0.314$ is written (see the top part of Fig. 9 in Ref. [18]). These misprints are not repeated in the bottom part of Fig. 9 where the correct formula is written for the energy dependence of the effective-range function $K_0(E_{lab})$. Furthermore, a reasonable description of the experimental dependence $K_0(E_{lab})$ is shown in the same figure in the area $E_{lab} \leq 6$ MeV (i.e., $E_\alpha \leq 3$ MeV in c.m.s. frame) with the parameters (16). For the ^{8}Be ground state with the set (15) we receive the following values of the resonant energy, width, NVC and ANC:

$$\begin{aligned} E_0 &= 92.248 \text{ keV}, \quad \Gamma_0 = 5.122 \text{ eV}; \\ \tilde{G}_0^2 &= (0.5047 - i0.0001151) \text{ fm}; \\ |\text{ ANC }| &= |\ C_0\ | = 0.001615 \text{ fm}^{-1/2}. \end{aligned} \quad (17)$$

All the results in Eqs. (17) have complex values due to the pole energy complexity. The resonance energy and width in Eqs. (17) agree well with Eq. (14). The value of $\tilde{G}_0^2$ is almost real because of the resonance pole proximity to the real energy axis. Using the expression (10) for narrow resonances we obtain $|\ C_0\ | = 0.001615$ fm$^{-1/2}$, which actually coincides with the value in Eqs. (17). This fact is quite natural for a resonance with such a small Γ_0. When Γ_0 is very small and energy E_0 is situated not far from a threshold, one does not need to use another method (for example, the SMP) because the related results are practically the same as those for the explicit expression (10).

IV. THE EXCITED ^{8}Be: THE d WAVE STATE RESONANCE IN THE $\alpha\,\alpha$ SCATTERING

Along with a description of the ^{8}Be ground state, the survey in Ref. [15] gives some features of the first excited state of the ^{8}Be nucleus ($J = 2^+$). The amplitude for the $\alpha\alpha$ scattering in the d wave has a pole at a cms energy of $E_\alpha = E_{\rm cm} = E_2 - i\Gamma_2/2$ (see Table (8.11) in Ref. [15] and notes to it on p. 184). In the laboratory frame, the

respective energy is $E_{lab} = 2E_{\rm cm}(E_{\rm cm} = k^2/2\mu$, where k is the relative momentum of colliding α particles and μ is the reduced mass of the $\alpha\alpha$ system). The weighted-mean values of the real part of the energy at the pole, $E_2 \pm \Delta E_2$, and of the resonance width, $\Gamma_2 \pm \Delta\Gamma_2$, are given in Ref. [15] along with the respective inaccuracies:

$$E_2 = 3.03 \pm 0.01 \text{ MeV}, \;\; \Gamma_2 = 1.49 \pm 0.02 \text{ MeV}. \quad (18)$$

These values are found from later results on the yields of the reactions ^{9}Be(p,d) and ^{9}Be(d,t) (see also Ref. [24]). Similar results of analyses performed for various reactions by various groups of authors at various times are also given in Ref. [15] [see the references in Table (8.9)]. Those results differ only in the value of the uncertainty in the resonance width, $\Delta\Gamma_2 = 0.015$ MeV. From Table (8.11) in Ref. [15], we find the ranges of mean values of E_2 and Γ_2 and the scatter of the uncertainties in them:

$$2.82 \le E_2 \le 3.18 \text{ MeV}, \;\; 10 \le \Delta \mathrm{E}_2 \le 200 \text{ keV};$$
$$1.20 \le \Gamma_2 \le 1.75 \text{ MeV}, \;\; 20 \le \Delta\Gamma_2 \le 300 \text{ keV}. \quad (19)$$

One can see that E_2 and Γ_2 and the boundaries of their variations are commensurate. It is noteworthy, however, that $\alpha\alpha$ scattering is not present among the reactions appearing in Table (8.11) from Ref. [15]. One of the objectives of the present study is to supplement the data quoted in Ref. [15] with data on $\alpha\alpha$ scattering by using the effective-range theory and S-matrix pole method. For the d wave resonance in question, we present the effective-range function in the form (11) of an expansion in the powers of k^2 up to k^4, and in the form which is the equivalent of the expansion in (12):

$$K_2(E_\alpha) = A_2 + B_{21}E_\alpha + B_{22}{E_\alpha}^2. \quad (20)$$

The calculated values of the functions $K_2(E_\alpha)$ and $\delta_2(E_\alpha)$ are highly sensitive to the position of the pole in the complex energy plane, and especially to the value E_2. The use of the values in Eq. (18) in fitting the parameters of the effective-range function $K_2(E_\alpha)$ along with the experimental value of the phase shift δ_2 from Ref. [21] (see Table II there) at the energy where the uncertainty $\Delta\delta_2$ is minimal distorts the shape of the energy dependence in relation to the experimental data. A partial-wave phase shift analysis of $\alpha\alpha$ scattering was performed more than 40 years ago (see references in Ref. [21]). The more recent publication by Warburton [25] contains information on the experimental dependence $\delta_2(E)$ in the form of a graph. There is virtually no difference between the data in Ref. [25] and the data on $\delta_2(E_\alpha)$ in Ref. [21] (Table 2 on p. 252), with the exception of several extra points in the region where $\delta_2 \le \pi/2$. The positions of these points fit well with the general character of the energy dependence of δ_2 in Ref. [21]. The phase shift δ_2 begins to manifest itself for $E_\alpha \ge 1.25$ MeV, the resonance lying completely in the region of convergence of $K_2(E_\alpha)$. Indeed, we find from Eq. (18) that $\mid E_2 - i\Gamma_2/2 \mid \approx 3$ MeV < 5 MeV. As soon as the considered resonance is broad enough, we apply both methods (the EFE and the SMP) to describe its characteristics. We take the experimental [22] resonance energy and width as

$$E_2 = 3.122 \text{ MeV}, \;\; \Gamma_2 = \;\; 1.513\text{MeV}. \quad (21)$$

First, we use the EFE method. The corresponding fitting effective-range function $K(E_\alpha)$ taken in the form (20) leads to $K(E_\alpha)$ parameters (with an energy in MeV):

$$A_2 = 0.0182 \text{ fm}^{-5}, \;\; B_{21} = -0.0056 \text{ fm}^{-5}\text{MeV}^{-1},$$
$$B_{22} = 0.0027 \text{ fm}^{-5}\text{MeV}^{-2}. \quad (22)$$

The set (22) corresponds to the parameters of Eq. (11):

$$a = -55.0 \text{ fm}^5, \;\; r = -0.1166 \text{ fm}^{-3}, \;\; P = 183.9 \text{ fm}^8. \quad (23)$$

The function $K_2(E_\alpha)$ fitting results are shown in Fig. 2 (a solid curve) where the experimental energy interval up to 11 MeV (in the c.m.s. frame) is considered. One can see that this EFE variant describes the experimental phase shift dependence on the energy only when $E_{lab} < 5$ MeV. We get the following results:

$$E_2 = 2.897 \text{ MeV}, \;\; \Gamma_2 = 1.470 \text{ MeV};$$
$$\tilde{G}_2^2 = (0.0137 - i0.0169) \text{ fm}; \quad (24)$$
$$\mid \text{ANC} \mid = \mid C_2 \mid = 0.3152 \text{ fm}^{-1/2}.$$

The resonant energy E_2 and the Γ_2 in Eqs. (24) are smaller than the experimental values given in Ref. [22] but the differences are not large. We also note that when using Eq. (10) (for a narrow resonance) $|C_2| = 0.3624$ fm$^{-1/2}$. The difference is not very big for such a broad resonance. To extend a good description of $K_2(E_\alpha)$ when $E_\alpha > 5$ MeV, we apply a Padé approximant, adding one more parameter to the effective-range function, which takes the form:

$$K_2(E_\alpha) = [a_0 + a_1E_\alpha + a_2{E_\alpha}^2]/(1 + b_1E_\alpha), \quad (25)$$

where

$$a_0 = 0.028 \text{ fm}^{-5}, \;\; a_1 = -0.009799 \text{ fm}^{-5}\text{MeV}^{-1}, \quad (26)$$
$$a_2 = 0.002549 \text{ fm}^{-5}\text{MeV}^{-2}, \;\; b_1 = -0.05122 \text{ MeV}^{-1}.$$

In Fig. 2 the dashed curve for Eq. (25) practically coincides with the solid curve at $E_\alpha \le 5$ Mev but also reproduces well the experimental points at $E_\alpha \ge 5$ Mev. We obtain the following results:

$$E_2 = 2.9380 \text{ MeV} \;\; \Gamma_2 = 1.2296 \text{ MeV};$$
$$\tilde{G}_2^2 = (0.0117 - i0.0136) \text{ fm}; \quad (27)$$
$$\mid \text{ANC} \mid = \mid C_2 \mid = 0.289 \text{ fm}^{-1/2}.$$

The differences between the related values in Eqs. (24) and (27) are not very large. One can see that the resonance energy and width are smaller than the experimental values (21).

We apply the S-matrix pole method to the excited 2^+ state. We use the experimental resonance energy (21) as a trial value to estimate the centered momentum value

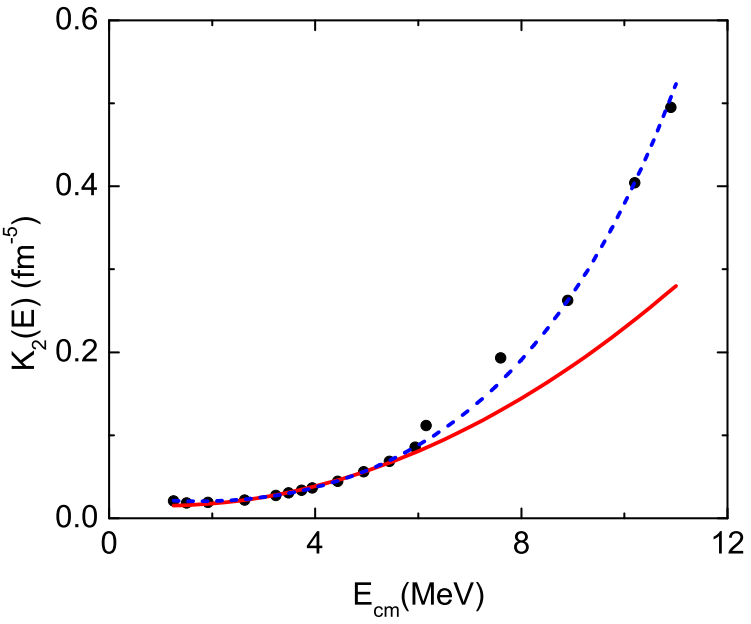

FIG. 2. Comparison of the fitted effective-range function for the d wave $\alpha\alpha$ elastic scattering (red color line) with the experimental values (points). The dashed line represents the result of fitting with the Padé approximant. The experimental data are taken from Ref. [21]. The energy is given in the c.m.s. frame.

for the series expansion (5). A very good phase shift fitting is achieved with four members up to $(k-k_s)^3$ in Eq. (5). The model describes the phase shift behavior as a function of the energy in the whole energy region (see Fig. 3) with the exception of the two experimental points with large uncertainties which obviously disagree with the general trend of the phase shift energy dependence. We find the following results:

$$E_2 = 2.916 \text{ MeV}, \;\; \Gamma_2 = 1.437 \text{ MeV};$$
$$\tilde{G}_2^2 = (0.01537 - i0.0145) \text{ fm}; \qquad (28)$$
$$\mid \text{ANC} \mid = \mid C_2 \mid = 0.3120 \text{ fm}^{-1/2}. \qquad (29)$$

We also perform a fitting parameters of the S-matrix pole method with the fixed values of the energy and width of the d state resonance, which are generally accepted as the experimental values and which are given in Eq. (21). In this case we also use the expansion up to terms $(k-k_s)^3$ of the nonresonant phase shift. However, as can be seen from Fig. 3 (dashed line), the agreement with the curve of the experimental points is worse compared with the previous method of the fitting, when we take the real and imaginary momentum of the resonance as the fitting parameters. We find the following results for the NVC and ANC:

$$\tilde{G}_2^2 = (0.0160 - i0.0067) \text{ fm};$$
$$\mid \text{ANC} \mid = \mid C_2 \mid = 0.2914 \text{ fm}^{-1/2}. \qquad (30)$$

One can see from Eqs. (24), (27), (28) and (30) that the results are very sensitive to the method used for fitting. In spite of the good phase shift fitting, we find that the different low-energy approaches lead to quite different results. The resonant energy occurs to be especially changeable and varies. This means there is disagreement between the experimental energy dependence of the phase shift on the one hand and the resonance complex energy on the other. So we recommend refining the phase shift for $\alpha\alpha$ scattering. Nevertheless some estimations of the ANC found in this work can be used in astrophysics.

V. ^{16}O BOUND STATES PROPERTIES FROM THE α^{12}C SCATTERING PHASE SHIFTS

The SMP is applicable to resonances but not to bound states. So in this section we apply the effective-range theory and the Padé approximant. For all considered ^{16}O bound states, the fitted effective-range function is quite well reproduced when the expansion (EFE) is limited to the expression (11) or (20) (with the dependence on k^2 or $E_{\rm cm}$). For the ground state $J^\pi = 0^+$, the fitting curve for $K_0(E_{\rm cm})$ is almost linear with the set of parameters in Eq. (12):

$$a_0 = -0.000328\,\text{fm}^{-1}; \;\; a_1 = 0.019450\,\text{fm}^{-1}\text{MeV}^{-1};$$
$$a_2 = 0.000171936\,\text{fm}^{-1}\text{MeV}^{-2}. \qquad (31)$$

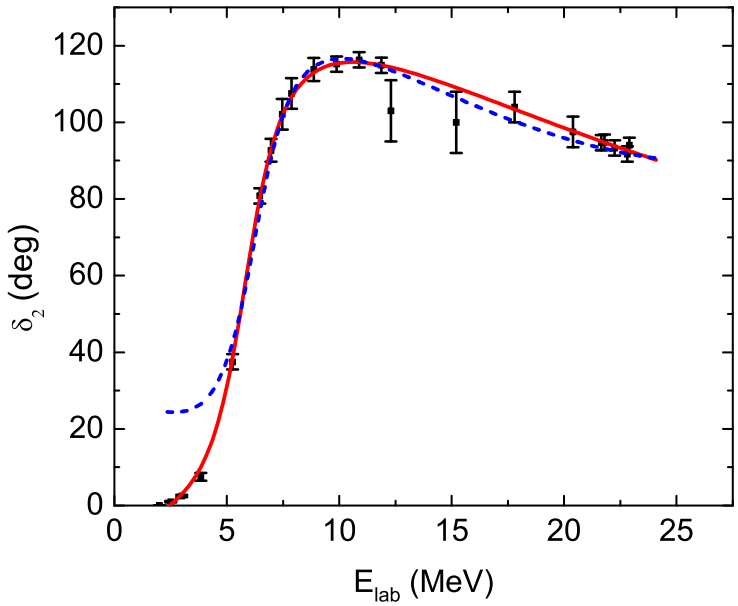

FIG. 3. Comparison of the fitted phase shifts for the d wave $\alpha\alpha$ elastic scattering obtained by the S-matrix pole method with the experimental data taken from Ref. [21]. The solid line shows when the complex momentum k_r of the resonance is added to the parameters of fitting. The dashed line indicates when the energy and width of resonance is fixed as in Eq. (21). The energy is given in the laboratory frame.

The curve $K_0(E_{\rm cm})$ crosses the point corresponding to the ground state at the experimental binding energy $\varepsilon = 7.162$ MeV. The fitting parameters (31) lead to the following results:

$$\tilde{G}_0^2 = 5.197 \text{ fm}; \;\; \text{ANC} = 20.33 \text{ fm}^{-1/2}. \qquad (32)$$

For the excited state $J^\pi = 1^-$ with the binding energy $\varepsilon_1 = 0.045$ MeV, we find the following fitting set of parameters:

$$a_0 = -0.00001773\,\text{fm}^{-3}; \;\; a_1 = 0.02930\,\text{fm}^3\text{MeV}^{-1};$$
$$a_2 = 0.003321\,\text{fm}^{-3}\text{MeV}^{-2}. \qquad (33)$$

The curve $K_1(E_{\rm cm})$ crosses the point $E_{\rm cm} = -0.045$ Mev corresponding to the excited 1^- state almost at the the point corresponding to the experimental binding energy and describes the experimental points quite well. The set of parameters (33) leads to the following results:

$$\tilde{G}_1^2 = 0.03584\,\text{fm}; \;\; \text{ANC} = 1.032 \times 10^{14}\,\text{fm}^{-1/2}. \qquad (34)$$

The very large ANC value for 1^- state is due to the small binding energy for this subthreshold level. For the excited state $J^\pi = 2^+$ with the binding energy $\varepsilon_1 = 0.245$ MeV, we find the following fitting set of parameters:

$$a_0 = -0.0004657 \text{ fm}^{-5}; \;\; a_1 = 0.00993814\,\text{fm}^{-5}\text{MeV}^{-1};$$
$$a_2 = 0.007050 \text{ fm}^{-5}\text{MeV}^{-2}. \qquad (35)$$

The curve $K_2(E_{cm})$ crosses the point corresponding to the excited 2^+ state almost at the point $E_{cm} = -0.245$ MeV, corresponding to the experimental binding energy, and describes the experimental points reasonably well. The set of parameters (35) leads to the following results:

$$\tilde{G}_2^2 = 0.001183\,\text{fm}; \;\; \text{ANC} = 21060\,\text{fm}^{-1/2}. \qquad (36)$$

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VI. ^{16}O RESONANT-STATE PROPERTIES FROM THE α^{12}C SCATTERING PHASE SHIFTS FOUND USING THE PADÉ APPROXIMANT FOR THE EFFECTIVE-RANGE FUNCTION

In our paper [5] we find that the effective-range method (EFE) is not able to reproduce properly the widths of the ^{16}O resonances while the S-matrix pole method can give reasonable results. In Ref. [5] we conclude that the SMP is successful because the central point k_s for expansion (5) is situated just near the resonance pole. In this section of the present paper we try to improve this situation by applying the Padé-approximant for the effective-range function instead of the polynomial expansion (EFE), taking into account the fact that the Padé approximant better reproduces the phase shift energy dependence. To do this, we study the resonances for the states with $J^\pi = 1^-$ and 3^- where the widths are bigger compared with other

states (see Table III in Ref. [5]). The SMP fits the experimental phase shifts for these states quite well. Fig. 3 of Ref. [5] shows that the resulting curves actually cross the experimental points. In Table I we compare the respective results obtained by the SMP and Padé approximant. Again we see that the results are not very different.

VII. CONCLUSION

In the present paper we continue to study the resonant states of light nuclei. Concretely we consider ^{8}Be in the 0^+ and 2^+ states and ^{16}O in the 1^- and 3^- states using three different low-energy methods: the effective-range expansion, the Padé approximant and the S-matrix pole method to calculate ANCs. We use as an input the phase shift energy behavior for the $\alpha\alpha$ and α^{12}C scattering borrowed from the literature. The SMP method is not applicable to bound states, so the EFE is used in the present paper to obtain the ANC for the three ^{16}O bound states: one ground ($\varepsilon = 7.162\,\mathrm{MeV}$, $\mathrm{J}^\pi = 0^+$) and two excited ($\varepsilon = 0.245$, MeV, $\mathrm{J}^\pi = 2^+$) and ($\varepsilon = 0.045$ MeV, $\mathrm{J}^\pi = 1^-$). All the nuclei considered are very important in astrophysics. But until now their ANC (at least for resonance states) have not been estimated theoretically. We emphasize that our methods for finding the resonant states ANC allow us to normalize the Gamov wave function which is quite difficult, especially in the presence of the Coulomb interaction. All the methods considered here are based on the phase shift analysis and on the analytical continuation of elastic scattering amplitudes (renormalized due to the Coulomb interaction) to nonphysical energy region. The results of this work show the contradiction between the resonance energy and the phase shift energy dependence. As the phase shift data for $\alpha\alpha$ scattering is about 40 years old, we recommend remeasuring. From our calculations we draw the following conclusions.

The ^{8}Be ground 0^+ resonant state is so narrow that the results are not sensitive to the model applied. In this case, a simple expression (10) can be used for the ANC, which is developed in Ref. [14].

The resonance for the ^{8}Be excited 2^+ state is broad enough to be more sensitive to the model used. We find in Ref. [5] that the EFE does not reproduce the Γ value while the SMP leads to a reasonable result for Γ. We also find that the application of the Padé-approximant instead of the EFE shows a better agreement with the SMP results. In this case, adding another parameter to introduce the pole into the effective-range function (zero for a partial scattering amplitude) leads to a much better description of $K_2(E_\alpha)$ in a larger energy area.

We show that applying the Padé-approximant to the ^{16}O in 1^- and 3^- states resonances also improves the agreement with the SMP results published in Ref. [5]. So in a way effective-range methods are 'rehabilitated' in the present paper after increasing the number of the fitting parameters and introducing a pole. In spite of this, the SMP is a better method for describing properties for broad resonances. A very good description of the experimental phase shift data is shown in Fig. 3 of our previous paper [5].

The ^{16}O bound states are also studied. We find very large differences in the ANC depending on the binding energy. The resonance near the threshold has a very big ANC due to the Γ-function in its definition (9). We note that in Ref. [27] the authors calculate the ANC for the first 2^+ excited state of ^{16}O. They choose a nuclear Gaussian potential, which reads $V(r) = -112.3319\exp(-r^2/2.82)\,\mathrm{MeV}$, where r is the distance between the clusters in fm, and the screened Coulomb potential is $e^2\mathrm{erf}(r/2.5)/r$, where $\mathrm{erf}(x)$ is the error function. This potential has a bound state when $E_\alpha = -245.0$ keV. Numerically, the authors find the ANC $= 1.384\times10^5\,\mathrm{fm}^{-1/2}$. This value can be compared with our result ANC$=1.0323\times10^4\,\mathrm{fm}^{-1/2}$.

The results of this paper can be used for solving nuclear astrophysical problems and may be applied to the theory of nuclear reactions using Feynman diagrams to describe the reaction mechanisms.

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TABLE I. States, energies and widths of ^{16}O nucleus levels above the α^{12}C threshold from our fits, as well as the corresponding values of the calculated NVC and ANC from the elastic α^{12}C scattering phase shifts [26]. Four terms of Eq. (5) are used for fitting. The energies of the resonances are given in the center-of-mass system of α^{12}C

.

Method	J^π	E_r (MeV)	Γ (MeV)	$\tilde{G}_l^2$ (fm)	C_l (fm$^{-1/2}$)	$\mid C_l \mid$ (fm$^{-1/2}$)
SMP	1^-	2.364	0.356	$4.970 - i1.797$	$0.1530 - i0.1032$	0.185
Padé	1^-	2.362	0.347	$4.862 - i1.789$	$0.1516 - i0.1015$	0.182
SMP	3^-	4.214	0.812	$0.276 - i0.142$	$-0.233 - i0.020$	0.234
Padé	3^-	4.239	0.786	$0.270 - i0.112$	$-0.228 - i0.029$	0.230

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